Turbulent flow

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Contents

1 Turbulence modeling

We will be assume constant density and viscosity of fluid. We also assume that no thermal interaction of the fluid with the solid boundary. In this way only continuity and momentum equations describe fluid velocity (u, v, w) and pressure p distribution (Navier -Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\, \nabla p + \nu \Delta \mathbf{v} \qquad \text{momentum equation} \tag{1}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{continuity equation–conservation of mass} \tag{2}$$

Equations (1),(2) are subjected to no–slip boundary condition at the walls and knows inlet and exit conditions.

Both laminar and turbulent flows satisfy (1),(2). For laminar flow, where there are no random fluctuations, we can sometimes solve them for a variety of geometries, like flow in pipe (see lecture n5–viscous–flow). **Most flows encountered in engineering practice are turbulent**. This is particularly true for pipe flows, so it is essential at this time to introduce a few very fundamental notions that will lead us to a better physical understanding of the friction factors, and hence the pressure losses, in such flows. It is useful to begin by recalling the difference in the nature of velocity profiles between laminar and turbulent flow in duct. This is depicted in Fig.1. The parabolic profile of part (a) corresponds to a fully-developed Poiseuille flow for which it can be seen that

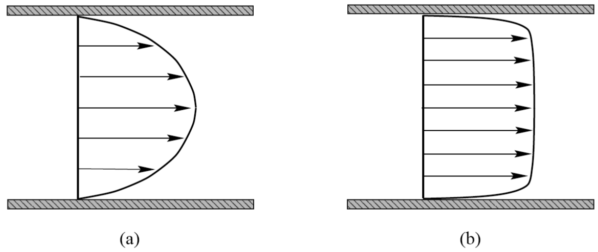

Figure 1: Comparison of laminar and turbulent velocity profiles in duct; a) laminar, b) turbulent

the velocity gradient at the wall, and hence also the wall shear stress, τ_w, is not so large as in the turbulent case of part (b) representing the (time) mean flow for fully-developed turbulence. The region very close to the wall exhibits a nearly linear velocity profile in the turbulent case, and is completely dominated by viscous effects. This inner layer is termed as the ***viscous sublayer***; velocity varies linearly with distance from the wall. The so-called ***”outer region”*** or called also as *inertial sublayer*, shows nearly constant velocity with distance from the wall. But we recognize that this outer layer velocity cannot satisfy the no–slip condition at the wall, and at the same time the inner (linear) profile which *does* satisfy no slip condition ($\mathbf{u} = 0$) will not correctly asymptote to the outer solution. This suggests that a third solution,***”overlap layer”***,is needed to match these two results.
The thickness of the viscous sublayer is very small (typically, much less than 1 percent of the pipe diameter), but this thin layer next to the wall plays a dominant role on characteristics because of large velocity gradients it involves.

For turbulent flow, because of the fluctuations, every velocity and pressure term in (1),(2) is a rapidly varying random function of time and space.

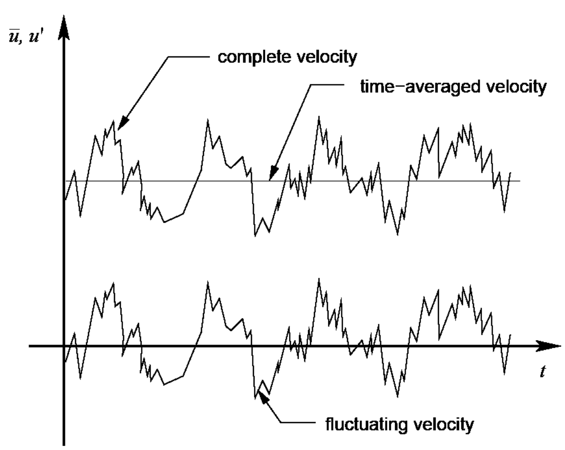

Figure 2: Graphical depiction of components of Reynolds decomposition

At present our mathematics cannot handle such instantaneous fluctuating variables. No single pair of random functions $u(x,y,z,t)$ and $p(x,y,z,t)$ is known to be a solution to (1),(2). As engineers we are interested more in the average, mean values of velocity, pressure, shear stress,etc.. This approach led Osborne Reynolds in 1895 to the decomposition, now known as the Reynolds decomposition

$$u(x,y,z,t) = \overline{u} + u'(x,y,z,t), \quad v(x,y,z,t) = \overline{v} + v', \quad w(x,y,z,t) = \overline{w} + w', \quad p(x,y,z,t) = \overline{p} + p' \tag{3}$$

In equation (3) the bar ("-") denotes a time average,and prime (" $'$ ") indicates fluctuation about the time–averaged, mean quantity. Formally, time average is defined by

$$\overline{u} = \lim_{T\to\infty} \frac{1}{T} \int_0^T u dt \tag{4}$$

The fluctuation u' is defined as the deviation of u from its average value $u' = u - \overline{u}$ and by definition a fluctuation has zero mean value.

$$\overline{u'} = \lim_{T\to\infty} \frac{1}{T} \int_0^T (u - \overline{u}) dt = \overline{u} - \overline{u} = 0 \tag{5}$$

For a time average to make sense, the integrals (4) and (**??**) have to be independent form initial time $t = 0$. It means that the mean flow has to be statistically steady:

$$\frac{\partial \overline{u}}{\partial t} = 0. \tag{6}$$

The mean square of a fluctuation is not zero $\overline{u'^2} > 0$ and is a measure of the *intensity of the turbulence*:

$$\overline{u'^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^T u'^2 dt \neq 0 \tag{7}$$

Also in general the mean fluctuation products such as $\overline{u'v'}$, $\overline{u'p'}$ is not equal zero in a typical turbulent flow. Substitute (3) into Navier –Stokes equations (1),(2), and take the time mean of each equation. The continuity relation reduces to

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{8}$$

which is no different from a laminar continuity relation (2). We can aslso show easily that

$$\frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} + \frac{\partial \overline{w'}}{\partial z} = 0. \tag{9}$$

So (8) and (9) tell us that the mean and fluctuating parts of velocity field each satisfy the continuity condition.
More interesting is what happens to the equation for the momentum (1). Let us write the the first u–component of this equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{10}$$

The left side of equation (10) (material derivative=du/dt), using the continuity equation $\nabla \cdot \mathbf{u} = 0$, one may rewrite as

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \tag{11}$$

Inserting to the equation (10), (11) $u = \overline{u} + u', v = \overline{v} + v', w = \overline{w} + w'$, after time averaging, will contain mean values plus three mean products, or correlations, of fluctuating velocities. The most important of these is the momentum relation in the mainstream, or x, direction, which takes the form

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial \overline{u}}{\partial x} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial \overline{u}}{\partial z} - \rho\overline{u'w'}\right) \tag{12}$$

The three correlation terms $-\rho\overline{u'^2}$,$-\rho\overline{u'v'}$, $-\rho\overline{u'w'}$ are called *turbulent stresses*,τ_{turb} because they have the same dimensions and occur right alongside the newtonian (laminar) stress terms $\tau_{lam} = \mu\frac{\partial u}{\partial x}$, etc. Actually, they are convective acceleration terms (which is why the density appears), not stresses, but they have the mathematical effect of stress and are so termed almost universally in the literature. **The turbulent stresses are unknown a priori** and must be related by experiment to geometry and flow conditions. The problem, how relate the $-\rho\overline{u'^2}$,$-\rho\overline{u'v'}$, $-\rho\overline{u'w'}$ to the the mean velocities $\overline{u}$,$\overline{v}$,$\overline{w}$ has occupied a lot of people for a long time now.

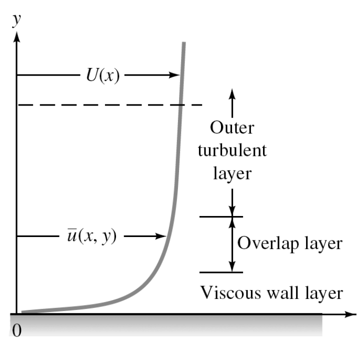

Figure 3: Typical velocity distributions in turbulent flow near wall

In pipe and boundary-layer flow, the stress $\rho\overline{u'v'}$ associated with direction y normal to the wall is dominant, and we can approximate with excellent accuracy a simpler streamwise momentum equation

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial \tau}{\partial y} \tag{13}$$

where

$$\tau = \mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'} = \tau_{lam} + \tau_{turb} \tag{14}$$

In the outer layer τ_{turb} is two or three orders of magnitude greater than τ_{lam}, and vice versa in the wall layer. These experimental facts enable us to use a crude but very effective model for the velocity distribution $\overline{u}(y)$ across a turbulent wall layer.

2 Velocity profiles: the inner, outer, and overlap layers

We have seen in Fig. 3 that there are three regions in turbulent flow near a wall:

1. Wall layer: Viscous shear dominates.
2. Outer layer: Turbulent shear dominates.
3. Overlap layer: Both types of shear are important.

Let τ_w be the wall shear stress, and let δ and U represent the thickness and velocity at the edge of the outer layer, $y = \delta$. For the wall layer, Prandtl deduced in 1930 that $\overline{u}$ must be independent of the shear–layer thickness

$$\overline{u} = f(\mu, \tau_w, \rho, y) \tag{15}$$

By dimensional analysis, this is equivalent to

$$\overline{u} = \phi\left(\frac{y}{\nu}\left(\frac{\tau_w}{\rho}\right)^{1/2}\right)\left(\frac{\tau_w}{\rho}\right)^{1/2} \tag{16}$$

where ϕ is non-dimensional function.
The quantity $\left(\frac{\tau_w}{\rho}\right)^{1/2} = u^*$ is termed the ***friction velocity*** because it has dimension m/s, although it is not actually a flow velocity. Equation(16) can be rewrite in dimensionless form:

$$u^+ = \frac{\overline{u}}{u^*} = \phi(\frac{yu^*}{\nu}) \tag{17}$$

Equation (17) is called the ***law of the wall***, and it is found to satisfactorily correlate with experimental data for smooth surface for $0 \leq yu^*/\nu \leq 5$. Therefor, the thickness of the viscous sublayer is roughly

$$\boxed{y = \delta_{sublayer} = \frac{5\nu}{u^*}} \tag{18}$$

The viscous sublayer gets thinner as the mean velocity increases. Consequently, the velocity profile becomes nearly flat and that the velocity distribution becomes more uniform at very high Reynolds number (very low viscosity).
From experiment it is known that function $\phi\left(\frac{yu^*}{\nu}\right) = \frac{yu^*}{\nu}$. From this fact follows the linear viscous relation

$$\boxed{u^+ = \frac{\overline{u}}{u^*} = \frac{yu*}{\nu} = y^+} \tag{19}$$

The quantity $\frac{\nu}{u^*}$ has dimension of length and is called the ***viscous length***; it is used to nondimensionalize the distance y from the surface.

Von Karman in 1933 deduced that $\overline{u}$ in the outer layer is independent of molecular viscosity, but its deviation from the stream velocity U must depend on the layer thickness δ and the other properties

$$(U_{max} - \overline{u})_{outer} = g(\delta, \tau_w, \rho, y) \tag{20}$$

Again, by dimensional analysis we rewrite this as

$$\frac{U_{max} - \overline{u}}{u^*} = \Phi(\frac{y}{\delta}) \tag{21}$$

where u^* is friction velocity. The deviation of velocity from the centerline value $U_{max} - u$ is called the ***velocity defect*** or retardation of the flow due to wall effects. Equation (21) is called the ***velocity-defect law*** for the outer layer.
Both the wall law 19) and the defect law 21) are found to be accurate for a wide variety of experimental turbulent duct and boundary-layer flows.
They are different in form, yet they must overlap smoothly in the intermediate layer. In 1937 C. B. Millikan showed that this can be true only if the overlap-layer velocity varies logarithmically with y:

$$\frac{\overline{u}}{u^*} = \frac{1}{\kappa}\ln\frac{yu*}{\nu} + B, \quad \text{or} \quad u^+ = \frac{1}{\kappa}\ln y^+ + B \tag{22}$$

Over the full range of turbulent smooth wall flows, the dimensionless constants κ and B are found to have the approximate values $\kappa = 0.41$ and $B = 5.0$. Equation 22) is called the ***logarithmic-overlap layer***. The results are summarize in figure 4.

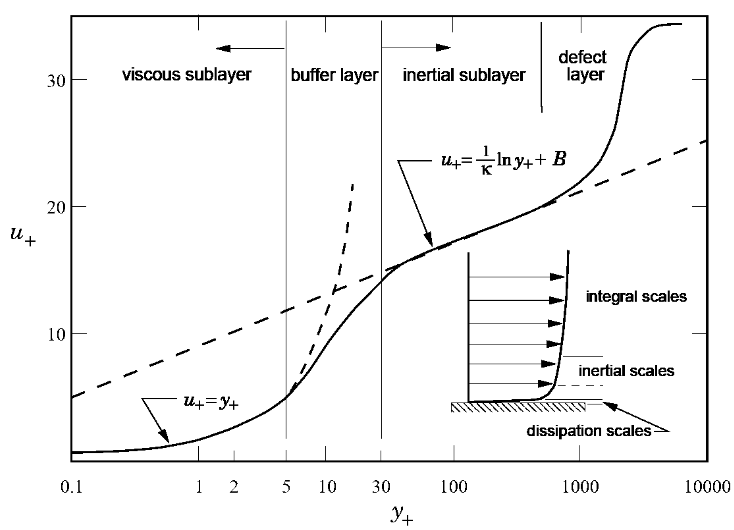

Figure 4: Experimental verification of the inner-, outer-, and overlap- layer laws relating velocity profiles in turbulent wall flow.

2.1 Turbulent-Flow Solution

Assume that (22) correlates the local mean velocity u(r) across the pipe. Setting $y = R - r$ we have

$$\frac{\overline{u}(r)}{u^*} = \frac{1}{\kappa} \ln \frac{(R-r)u^*}{\nu} + B \tag{23}$$

Compute the average velocity V from this profile

$$\begin{aligned} V = \frac{q}{\pi R^2} &= \frac{1}{\pi R^2} \int_0^R u^* \left[\frac{1}{\kappa} \ln \frac{(R-r)u^*}{\nu} + B \right] 2\pi r dr \\ &= \frac{1}{2}\, u^* \left(\frac{2}{k}\, \ln \frac{Ru^*}{\nu} + 2B - \frac{3}{k} \right) \end{aligned} \tag{24}$$

The process of integration the above integral, eq. (24) is rather laborious. If you are not patient enough the easiest way to calculate the above integral (eq. (24) is the usage of computer program for algebraic, symbolic manipulation like ”MATHEMATICA”. Introducing $k = 0.41.$ and $B = 5.0$ we obtain, numerically,

$$\frac{V}{u^*} \approx 2.44 \ln \frac{Ru^*}{\nu} + 1.34 \tag{25}$$

What it is here exciting that we can directly related formula (25) to the Darcy friction factor f. Let us recalled that in lecture ”n5–viscous–flow” we related the pressure drop to the shear stress as follow:

$$\frac{\Delta p}{l} = \frac{2\tau}{r} \tag{26}$$

We concluded that the shear stress had to be a linear function of the r–variable $\tau = Cr$ and the constant C can be express by the wall share $C = 2\tau_w/D$. So

$$\tau = \frac{2\tau_w r}{D} \tag{27}$$

and

$$\Delta p = \frac{4l\,\tau_w}{D} \tag{28}$$

The Darcy -Weisbach law says:

$$\boxed{h_L = \frac{\Delta p}{\rho g} = f\, \frac{L}{D}\, \frac{v^2}{2g}} \tag{29}$$

and from this we have

$$f = \frac{\Delta p}{\frac{1}{2}\rho V^2} \frac{D}{L} \tag{30}$$

By substituting the pressure drop expressed by the wall shear (28) we obtain an alternate expression for the friction factor as a dimensional wall shear stress

$$\boxed{f = \frac{8\tau_w}{\rho V^2} \quad \text{or} \quad f = 8\left(\frac{u^*}{V}\right)^2 \quad \text{and} \quad \tau_w = \frac{1}{4} f \frac{\rho V^2}{2}} \tag{31}$$

Now, the left side of the equation (25) can be express as

$$\frac{V}{u^*} = \left(\frac{V^2}{\frac{\tau_w}{\rho}}\right)^{1/2} = \left(\frac{8}{f}\right)^{1/2} \tag{32}$$

The argument of the logarithm in (25) may by express equivalently

$$\frac{Ru^*}{\nu} = \frac{\frac{1}{2}D\,Vu^*}{\nu V} = \frac{1}{2}\,Re\,\left(\frac{f}{8}\right)^{1/2} \tag{33}$$

Introducing (33) into Eq. (25), changing to a base–10 logarithm, and rearranging, we obtain

$$\frac{1}{f^{1/2}} \approx 1.99\,\log(Re\,f^{1/2}) - 1.02 \tag{34}$$

Summarizing, by simply computing the mean velocity from the logarithmic–low correlation, we obtain a relation between the friction and Reynolds number for turbulent pipe flow. Prandtl derived Eq. (34)in 1995 and adjusted the constants slightly to fit friction data better

$$\boxed{\frac{1}{f^{1/2}} = 2.0\,\log(Re\,f^{1/2}) - 0.8} \tag{35}$$

Equation (35) is cumbersome to solve if *Re* is known and f is wanted. As we already have known there are many aproximations in the literature from which f can be computed explixity form *Re*, for example well–known Blasius formula

$$\boxed{f = (100Re)^{-1/4}} \tag{36}$$

The maximum velocity in turbulent pipe flow is given by (23), evaluated at r=0

$$\frac{U_{max}}{u^*} = \frac{1}{\kappa}\ln\frac{(R)u^*}{\nu} + B \tag{37}$$

Subtracting equation (37) form (25)one obtain

$$\frac{V - U_{max}}{u^*} = 3.66 \tag{38}$$

Using (32) we obtain the formula relating mean velocity to maximum velocity

$$U_{max} \approx V(1 + 1.33\,\sqrt{f}) \tag{39}$$

2.2 Power-law velocity profile

Numerous other empirical velocity profiles exist for turbulent pipe flow. Among those, the simplest and the best known is the *power-law velocity* profile express as

$$\frac{\overline{u}}{U_{max}} = \left(\frac{y}{R}\right)^{1/n}, \quad \text{or} \quad \frac{\overline{u}}{U_{max}} = \left(1 - \frac{r}{R}\right)^{1/n} \tag{40}$$

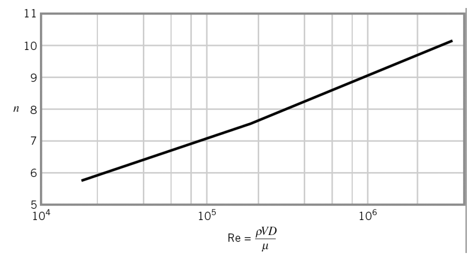

Figure 5: Exponent,n, for power-law velocity profiles.

where the exponent n is a constant whose value depends on the Reynolds number. The value n increases with increasing Reynolds number. The value $n = 7$ generally approximates many flows in practice, giving rise to the term *one-seventh power-law velocity profile*. Typical turbulent velocity profiles based on power–law representation are shown in Fig. (6)

Note from (6) that the turbulent profiles are much ”flatter than the laminar profile and that this flatness increses with Reynolds number.

Example 1. *Water at $20^0 C$ ($\rho = 998\ kg/m^3$, and $\nu = 1 \cdot 10^{-6}\ m^2/s$ flows through a horizontal pipe of $D = 0.1\ m$ diameter with a flowrate of $q = 4 \cdot 10^{-2}\ m^3/s$ and a pressure gradient of $\frac{\Delta p}{l} = 2.59 kPa/m$.*

1. *Determine the approximate thickness of the viscous sublayer.*
2. *Determine the approximate Darcy friction coefficient f*
3. *Determine the approximate centerline velocity, U_{max} by two methods: one using the power–law velocity profile theory and the second using overlap-layer velocity profile theory (eq. (40))*
4. *Determine the ratio of the turbulent to laminar shear stress, τ_{turb}/τ_{lam} at a point midway between the centerline and the pipe wall (i.e., at $r = 0.025\ m$)*

SOLUTION.

(1) *According to Eq. (18), the thickness of the viscous sublayer, is approximately*

$$\delta_s = 5\ \frac{\nu}{u^*} \qquad where \qquad u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

The wall shear stress can be obtained from the pressure drop data $\frac{\Delta p}{l} = 2.59 \cdot 10^3\ Pa$ (see equation (28)):

$$\tau_w = \frac{D}{4}\left(\frac{\Delta p}{l}\right) = 64.8\ N/m^2$$

Hence, $u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2} = 0.255\ m/s$.

The centerline velocity can be obtained from the average velocity and and the formula (39) or the

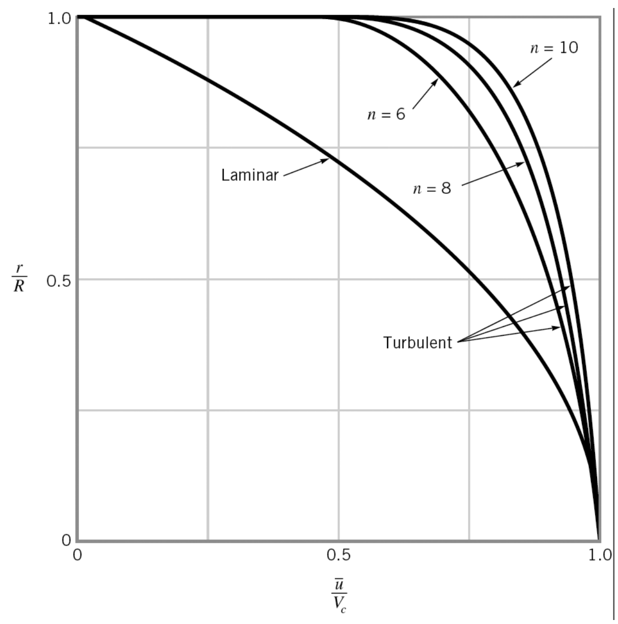

Figure 6: Typical laminar flow and turbulent flow velocity profiles.

assumption of a power–law velocity profile as follows. For this flow $V = q/A = 5.09$ m/s *The Reynolds number is*

$$Re = \frac{VD}{\nu} = 5.07 \cdot 10^5.$$

Thus, from Fig. (5) $n = 8.4$*, so that*

$$\frac{\overline{u}}{U_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{8.4}}$$

(2) *The friction coefficient can be evaluate from the formula Eq. (31)*

$$f = \frac{8\tau_w}{\rho V^2} = \frac{8 \cdot 64.8}{998.0 \cdot 5.04^2} = 0.02 \tag{41}$$

(3) *To determine the central velocity* U_{max}*, we must know the relationship between* V *and* U_{max}*. This can be obtained by integration of the power-law velocity profile (or using the formula (39), please compare the results yourself)*

$$q = AV = \int \overline{u} dA = U_{max} \int_0^R \left(1 - \frac{r}{R}\right)^{1/n} = 2\pi R^2 U_{max} \frac{n^2}{(n+1)(2n+1)}$$

Since $q = \pi R^2 V$ *we obtain*

$$\frac{V}{U_{max}} = \frac{2n^2}{(n+1)(2n+1)} \tag{42}$$

With $n = 8.4$ in the prsent case, this gives $U_{max} = 6.04\ m/s$.
***(4)** From Eq. (27) the shear stress at $r = 0.025\ m$ is*

$$\tau = \frac{2\tau_w r}{D} = \frac{2(64.6 \cdot 0.025)}{0.1} = 32.4\ \frac{\mathrm{N}}{m^2} \tag{43}$$

Shear stress is the sum of $\tau = \tau_{lam} + \tau_{turb}$. The laminar shear stress $\tau_{lam} = -\mu\ d\overline{u}/dr$ and turbulent share stress $\tau_{turb} = -\rho\overline{u'v'}$. From the power -law velocity profile (40) we obtain the gradient of the average velocity as

$$\frac{d\overline{u}}{dr} = \frac{-U_{max}}{nR}\left(1 - \frac{r}{R}\right)^{(1-n)/n} = -\frac{6.04}{8.4 \cdot 0.05}\left(1 - \frac{0.025}{(}0.05)\right)^{(1-8.4)/8.4} = -26.5\ 1/s$$

Thus

$$\tau_{lam} = -\mu\frac{d\overline{u}}{dr} = -(\nu\rho)\frac{d\overline{u}}{dr} = -(1.0 \cdot 10^{-6}) \cdot 998 \cdot (-26.5) = 0.0266\ \mathrm{N}/m^2$$

The ratio of turbulent to laminar shear stress is given by

$$\frac{t_{turb}}{\tau_{lam}} = \frac{\tau - \tau_{lam}}{\tau_{lam}} = \frac{32.4 - 0.026}{0.026} = 1220$$

As expected, most of the shear stress at this location in the turbulent flow is due to the turbulent shear stress.

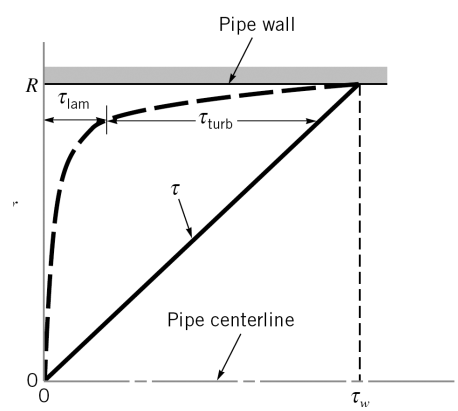

Figure 7: Shear stresses of turbulent flow in pipe. The shear stress τ in pipe is a linear function of the distance from the central line $\tau = Cr$. In a very narrow region near wall (the viscous sublayer) the laminar shear stress is dominant. The scale of on the figure is not correct. Typically the value of τ_{turb} is 100 to 1000 times greater than τ_{lam} in the outer region.

Example 2. *Water flows at $Re = 2.3 \times 10^4$ through a horizontal smooth pipe with $V = 0.807\ m/s$. Compare $\overline{u}$ for $y^+ = 5, 100$ and 500 from **a)** the logarithmic law and **b)** the power law to the measure data, that is the solid line in Fig. (4), $\rho = 998\ kg/m^3$, $\nu = 1 \cdot 10^{-6}\ m^2/s$.*
***Solution**.*

a)*The log law* $u^+ = 2.44\ln y^+ + 5$ *for* $40 \le y^+ \le 600$*, has to augmented for* $y^+ = 5$*, which is located in a laminar sublayer for which* $u^+ = y^+$*. Again,* $u^+ = \overline{u}/u^*$ *and* $y^+ = yu^*/\nu$*, where* $y = R - r$*. First we find* $u^* = (\tau_w/\rho)^{1/2}$ *where*

$$\tau_w = \rho V^2 f/8 \quad with \quad f = (100Re)^{-1/4} \quad gives \quad u^* = 0.0457\ m/s$$

and $\overline{u}$ *at three location can be calculated.*

b) *The power law for pipe flow can be written as* $u = U_{max}(y/R)^{1/n}$ *where* $n \approx 6.6$ *for* $Re = 2.3 \times 10^4$*. Now from* $V = 1/\pi R^2 \int_0^R \overline{u}\ 2\pi r dr$ *the centerline velocity* U_{max} *can be derived as*

$$U_{max} = \frac{(n+1)(2n+1)}{2n^2}\ V = 1.0\ m/s$$

Data comparison for turbulent velocity profiles

y^+	$(y/R)*100\%$	*Log law*	*Power law*	*Measurements*
–	%	$\overline{u}\ m/s$	$\overline{u}\ m/s$	m/s
5	0.8	0.229	0.478	0.229
100	15.3	0.741	0.752	0.754
500	76.4	0.921	0.960	0.983

3 Turbulence: some important thoughts

”Most flows occurring in nature and in engineering applications are turbulent. The boundary layer in the earth’s atmosphere is turbulent (except possibly in very stable conditions); jet streams in the upper troposphere are turbulent; cumulus clouds are in turbulent motion. The water currents below the surface of the oceans are turbulent; the Gulf Stream is a turbulent wall-jet kind of flow. The photosphere of the sun and the photospheres of similar stars are in turbulent motion; interstellar gas clouds (gaseous nebulae) are turbulent; the wake of the earth in the solar wind is presumably a turbulent wake. Boundary layers growing on aircraft wings are turbulent. Most combustion processes involve turbulence and even depend on it; the flow of natural gas and oil in pipelines is turbulent. Chemical engineers use turbulence to mix and homogenize fluid mixtures and to accelerate chemical reaction rates in liquids or gases. The flow of water in rivers and canals is turbulent; the wakes of ships, cars, submarines and aircraft are in turbulent motion. The study of turbulence is clearly an interdisciplinary activity, which has a very wide range of applications. In fluid dynamics laminar flow is the exception, not the rule: one must have small dimensions and high viscosities to encounter laminar flow.”
We have discussed turbulence without offering any actual definition for it, because the turbulence is resistant to precise defnition. Instead, we have mentioned the following properties.

- **High Reynolds number**. Turbulence occurs at high Reynolds number. Remember that we said that instability of laminar flow was one way that flows become turbulent; instability of a particular flow requires that the Reynolds number exceed come minimum value. The high Reynolds number also means that at the large scales of the flow, inertial forces dominate over viscous forces.

- **Randomness and disorder.** Turbulent flows exhibit a high degree of randomness and disorder, particularly at small scales. We describe these random fluctuations by decomposing the flow variables into mean and fluctuating parts, e.g. $u_i = \overline{u}_i + u'_i$.

- **Disparity in length scales**. A wide range of length scales are relevant in turbulent flows. The spread between the largest and smallest length scales in a flow increases with increasing Reynolds number. This makes computation extremely difficult.

- **Energy cascade**. In a turbulent flow, energy generally gets transferred from the large scales to the small scales in an inviscid fashion, then gets dissipated at the small scales by the action of viscosity. The presence of dissipation in turbulent flows is significant; even though the large flow scales may be essentially inviscid, there is viscous dissipation of kinetic energy going on in the flow, at the small scales.

- **Increased transport of momentum, scalars etc.** The random fluctuations of turbulent flows provide another mechanism by which quantities can get transferred from one portion of a flow to another. In a laminar flow, transport of momentum mostly occurs through viscosity (diffusion), while in a turbulent flow, transport can occur through the random motion embodied by the fluctuating terms. Think of a parallel flow, with all the (mean) velocity vectors pointing in the same direction (say, the x-direction). In the laminar case, momentum only gets transferred between layers by viscous drag. In the turbulent case, however, there can be velocity fluctuations in the y-direction to carry momentum across the mean streamlines.

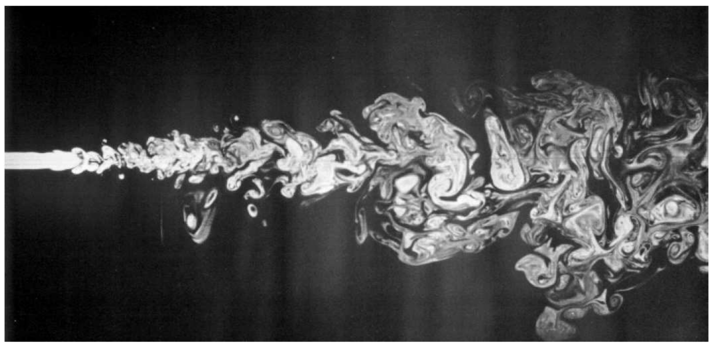

Figure 8: The scalar concentration of a turbulent water jet.

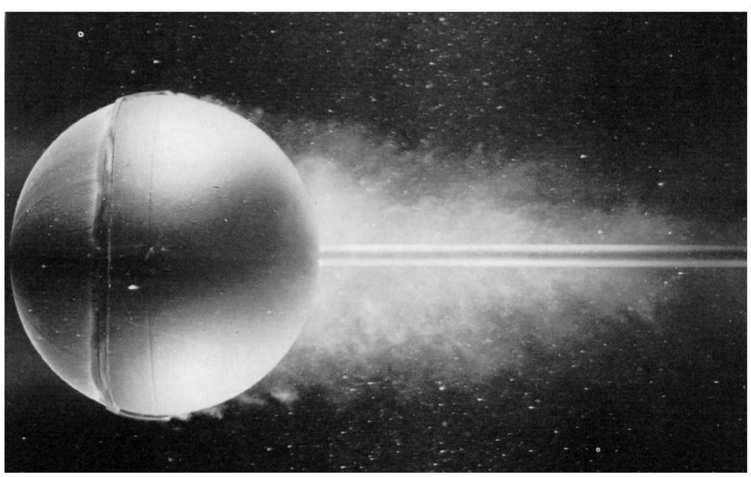

Figure 9: A turbulent boundary layer on a sphere

4 Problems

1. Air at 20^oC, $(\nu = 1.51 \cdot 10^{-5}\ m^2/s)$, $\rho = 1.205\ kg/m^3$, flow through a 14-cm-diameter tube under fully developed conditions. The centerline velocity is $U_{max} = 5\ m/s$. Estimate a) the friction velocity $u*$ assuming the logarithmic law (Eq. (**??**) - use the iteration) b) the wall stress τ_w, c) the average velocity V using the formula (25).
2. By analogy with laminar shear, $\tau = \mu du/dy$, T.V. Boussinesq in 1877 postulated that turbulent shear could also be related to mean-velocity gradient $\tau_{turb} = \varepsilon d\overline{u}/dy$, where ε is called the eddy viscosity and is much larger than molecular viscosity μ. If the logarithmic–overlap law, Eq. (**??**) is valid with $\tau \approx \tau_w$ show that $\varepsilon \approx k\rho u^* y$.
3. Water $(\rho = 998,\ \nu = 1 \cdot 10^{-6}\ m^2/s)$ flows in a 9–cm–diameter pipe under fully developed conditions. The centerline velocity $U_{max} = 10\ m/s$. Compute a) q, b) V (mean velocity), c)τ_w and d) Δp for a 100–m pipe length.